

# A Gauss-Seidel Iterative Thresholding Algorithm for $l_q$ Regularized Least Squares Regression <sup>☆</sup>

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## Abstract

In recent studies on sparse modeling,  $l_q$  ( $0 < q < 1$ ) regularized least squares regression ( $l_q$ LS) has received considerable attention due to its superiorities on sparsity-inducing and bias-reduction over the convex counterparts. In this paper, we propose a Gauss-Seidel iterative thresholding algorithm (called GAITA) for solution to this problem. Different from the classical iterative thresholding algorithms using the Jacobi updating rule, GAITA takes advantage of the Gauss-Seidel rule to update the coordinate coefficients. Under a mild condition, we can justify that the support set and sign of an arbitrary sequence generated by GAITA will converge within finite iterations. This convergence property together with the Kurdyka-Łojasiewicz property of ( $l_q$ LS) naturally yields the strong convergence of GAITA under the same condition as above, which is generally weaker than the condition for the convergence of the classical iterative thresholding algorithms. Furthermore, we demonstrate that GAITA converges to a local minimizer under certain additional conditions. A set of numerical experiments are provided to show the effectiveness, particularly, much faster convergence of GAITA as compared with the classical iterative thresholding algorithms.

**Keywords:**  $l_q$  regularized least squares regression, iterative thresholding algorithm, Gauss-Seidel, Jacobi, Kurdyka-Łojasiewicz inequality

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## 1. Introduction

In this paper, we consider the following  $l_q$  ( $0 < q < 1$ ) regularized least squares regression ( $l_q$ LS) problem

$$(l_q\text{LS}) \quad \min_{x \in \mathbf{R}^N} \left\{ T_\lambda(x) = \frac{1}{2} \|Ax - y\|_2^2 + \lambda \|x\|_q^q \right\}, \quad (1.1)$$

where  $\|x\|_q^q = \sum_{i=1}^N |x_i|^q$ ,  $N$  is the dimension of  $x$  and  $\lambda > 0$  is a regularization parameter. The ( $l_q$ LS) problem has attracted lots of attention in both scientific research and engineering practice, since it commonly has stronger sparsity-promoting ability and better bias-reduction property over the  $l_1$  case. Its typical applications include signal processing [12], [13], image processing [11], [23], synthetic aperture radar imaging [39], and machine learning [24].

One of the most important class of algorithms to solve the ( $l_q$ LS) problem is the iterative thresholding algorithm (ITA) [9], [38]. Compared with some other classes of algorithms such as the reweighted least squares (IRLS) minimization [16] and iterative reweighted  $l_1$ -minimization (IRL1) [10] algorithms, ITA generally has lower computational complexity for large scale problems [39], which triggered avid research activities of ITA in the past decade (see [8, 17, 38, 40]). The makeup of ITA comprises two steps: a gradient descent-type iteration for the least squares and a thresholding operator. To be detailed, for an arbitrary  $\mu > 0$ , the thresholding function (or proximity operator) for ( $l_q$ LS) can be defined as

$$Prox_{\mu, \lambda \|\cdot\|_q^q}(x) = \arg \min_{u \in \mathbf{R}^N} \left\{ \frac{\|x - u\|_2^2}{2\mu} + \lambda \|u\|_q^q \right\}. \quad (1.2)$$

Since  $\|\cdot\|_q^q$  is separable, computing  $Prox_{\mu, \lambda \|\cdot\|_q^q}$  can be reduced to solve several one-dimensional minimization problems, that is,

$$prox_{\mu, \lambda |\cdot|^q}(z) = \arg \min_{v \in \mathbf{R}} \left\{ \frac{|z - v|^2}{2\mu} + \lambda |v|^q \right\}, \quad (1.3)$$

and thus,

$$Prox_{\mu, \lambda \|\cdot\|_q^q}(x) = (prox_{\mu, \lambda |\cdot|^q}(x_1), \dots, prox_{\mu, \lambda |\cdot|^q}(x_N))^T. \quad (1.4)$$

For some  $q$ , such as  $\frac{1}{2}$  or  $\frac{2}{3}$ ,  $prox_{\mu, \lambda |\cdot|^q}(\cdot)$  can be analytically expressed [38]. While for other  $q \in (0, 1)$ , we can use an iterative scheme proposed by [26] to compute the operator  $prox_{\mu, \lambda |\cdot|^q}(\cdot)$ . All these make the thresholding operator achievable. Then, an efficient gradient-descent iteration for the un-regularized least squares problem ( $\lambda = 0$  in ( $l_q$ LS)) together with the aforementioned thresholding operator can derive a feasible scheme to solve ( $l_q$ LS).

### 1.1. Jacobi iteration and Gauss-Seidel iteration

As the thresholding operator depends only on  $q$ , the convergence of ITA depends heavily on the attributions of the gradient-descent type iteration. Landweber-type iteration, is a natural selection to solve the un-regularized least squares problems, since its feasibility has been sufficiently verified in many literatures (say, [22]). In the classical ITA [8, 17, 38], a Jacobi iteration strategy whose Landweber iteration rule is imposed on the variable  $x^n$ , is employed to derive the estimate. We denote such algorithm as **JAITA** henceforth. More specially, JAITA for  $(l_q\text{LS})$  can be described as:

$$x^{n+1} \in \text{Prox}_{\mu, \lambda \|\cdot\|_q^q}(x^n - \mu A^T(Ax^n - y)), \quad (1.5)$$

where  $\mu > 0$  is a step size parameter.

As a cousin of the Jacobi scheme, the Gauss-Seidel scheme is also widely used to build blocks for more complicated algorithms [34, 35, 36, 37]. Different from the Jacobi iteration that updates all the components simultaneously, the Gauss-Seidel iteration is a component-wise scheme. Generally speaking, the Gauss-Seidel iteration is faster than the corresponding Jacobi iteration [36], since it uses the latest updates at each iteration. The aim of this paper is to introduce the Gauss-Seidel scheme to solve  $(l_q\text{LS})$ . The core construction of the detailed Gauss-Seidel update rule is by a concrete representation of the thresholding function, which is derived by the most recent work [9].

According to [9],  $\text{prox}_{\lambda\mu, q}(\cdot)$  can be expressed as

$$\text{prox}_{\mu, \lambda|\cdot|^q}(z) = \begin{cases} (\cdot + \lambda\mu q \text{sgn}(\cdot)|\cdot|^{q-1})^{-1}(z), & \text{for } |z| \geq \tau_{\lambda\mu, q} \\ 0, & \text{for } |z| \leq \tau_{\lambda\mu, q} \end{cases} \quad (1.6)$$

for any  $z \in \mathbf{R}$  with

$$\tau_{\lambda\mu, q} = \frac{2-q}{2-2q} (2\lambda\mu(1-q))^{\frac{1}{2-q}}, \quad (1.7)$$

$$\eta_{\lambda\mu, q} = (2\lambda\mu(1-q))^{\frac{1}{2-q}}, \quad (1.8)$$

and the range of  $\text{prox}_{\mu, \lambda|\cdot|^q}$  is  $\{0\} \cup [\eta_{\lambda\mu, q}, \infty)$ ,  $\text{sgn}(\cdot)$  represents the sign function henceforth. When  $|z| \geq \tau_{\lambda\mu, q}$ , the relation  $\text{prox}_{\mu, \lambda|\cdot|^q}(z) = (\cdot + \lambda\mu q \text{sgn}(\cdot)|\cdot|^{q-1})^{-1}(z)$  means that  $\text{prox}_{\mu, \lambda|\cdot|^q}(z)$  satisfies the following equation

$$v + \lambda\mu q \cdot \text{sgn}(v)|v|^{q-1} = z.$$

Now we are in a position to present the proposed algorithm by utilizing the Gauss-Seidel iteration. Given the current estimate  $x^n$  and the step size  $\mu$ , at the next iteration, the  $i$ -th

coefficient is selected cyclically by

$$i = \begin{cases} N & \text{if } 0 \equiv (n+1) \bmod N \\ (n+1) \bmod N & \text{otherwise} \end{cases}. \quad (1.9)$$

We then derive a component-based update of the un-regularized least squares by

$$z_i^n = x_i^n - \mu A_i^T (Ax^n - y), \quad (1.10)$$

which together with the thresholding operator then yields a component-based update for ( $l_q$ LS)

as

$$x_i^{n+1} \in \arg \min_{v \in \mathbf{R}} \left\{ \frac{|z_i^n - v|^2}{2} + \lambda \mu |v|^q \right\} = \text{prox}_{\mu, \lambda|\cdot|^q}(z_i^n).$$

It can be seen from (1.6) that  $\text{prox}_{\mu, \lambda|\cdot|^q}$  is a set-valued operator. Therefore, motivated by [26], we select a particular single-valued operator of  $\text{prox}_{\mu, \lambda|\cdot|^q}$  and then update  $x_i^{n+1}$  according to the following scheme,

$$x_i^{n+1} = \mathcal{T}(z_i^n, x_i^n), \quad (1.11)$$

where

$$\mathcal{T}(z_i^n, x_i^n) = \begin{cases} \text{prox}_{\mu, \lambda|\cdot|^q}(z_i^n) & \text{if } |z_i^n| \neq \tau_{\lambda\mu, q} \\ \text{sgn}(z_i^n) \eta_{\lambda\mu, q} \mathbf{I}(x_i^n \neq 0), & \text{if } |z_i^n| = \tau_{\lambda\mu, q} \end{cases},$$

and  $\mathbf{I}(x_i^n \neq 0)$  denotes the indicator function, that is,

$$\mathbf{I}(x_i^n \neq 0) = \begin{cases} 1, & \text{if } x_i^n \neq 0 \\ 0, & \text{otherwise} \end{cases}.$$

While the other components of  $x^{n+1}$  are fixed, i.e.,

$$x_j^{n+1} = x_j^n, \text{ for } j \neq i. \quad (1.12)$$

For the sake of brevity, we denote in the rest of paper  $\tau_{\mu, q}$  and  $\eta_{\mu, q}$  to take the place of  $\tau_{\lambda\mu, q}$  and  $\eta_{\lambda\mu, q}$ , respectively. In summary, we can formulate the proposed algorithm as follows.

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Gauss-Seidel Iterative Thresholding Algorithm (GAITA)

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Initialize with  $x^0$ . Choose a step size  $\mu > 0$ , let  $n := 0$ .

Step 1. Calculate the index  $i$  according to (1.9);

Step 2. Calculate  $z_i^n$  according to (1.10);

Step 3. Update  $x_i^{n+1}$  via (1.11) and  $x_j^{n+1} = x_j^n$  for  $j \neq i$ ;

Step 4. Check the terminational rule. If yes, stop;

otherwise, let  $n := n + 1$ , go to Step 1.

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### 1.2. Why Gauss-Seidel?

It can be found in the last section that the main difference between JAITA and GAITA lies at whether the Landweber iteration is component-wise. Such a slight difference leads to a plausible assertion that the convergence of both algorithms are similar. To verify the authenticity of the above viewpoint, we conduct a set of experiments to the convergence of JAITA and GAITA. Interestingly, we find in this experiment that the convergence of the aforementioned two algorithms are totally different.

To be detailed, given a sparse signal  $x$  with dimension  $N = 500$  and sparsity  $k^* = 15$ , we considered the signal recovery problem through the observation  $y = Ax$ , where the original sparse signal  $x$  was generated randomly according to the standard Gaussian distribution, and  $A$  was of dimension  $m \times N = 250 \times 500$  with Gaussian  $\mathcal{N}(0, 1/250)$  i.i.d. entries and was preprocessed via column-normalization, i.e.,  $\|A_i\|_2 = 1$  for any  $i$ . We then applied GAITA and JAITA to the  $(l_q\text{LS})$  problem with two different  $q$ , that is,  $q = 1/2$  and  $2/3$ , respectively. In both cases, the thresholding functions can be analytically expressed as shown in [38] and [11], respectively, and thus the corresponding algorithms can be efficiently implemented. In both cases, we set  $\lambda = 0.001$ ,  $\mu = 0.95$  for both JAITA and GAITA. Moreover, the initial guesses were taken as 0 for all cases. The trends of the objective sequences in different cases are shown in Fig. 1.

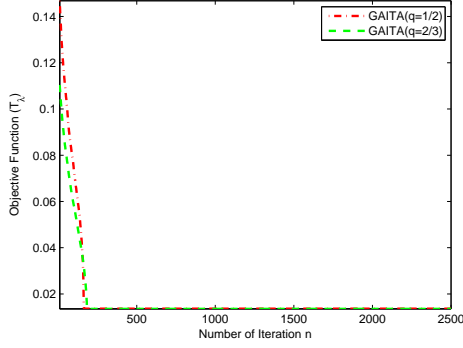
From Fig. 1, the objective sequences of JAITA diverge for both  $q = 1/2$  and  $2/3$ , while those of GAITA are definitely convergent. This means that there exists some  $\mu$  such that JAITA is divergent but GAITA is assuredly convergent, which significantly stimulates our research interests, since a large scope of  $\mu$  to guarantee the convergence essentially enlarges the applicable range of iterative thresholding-type algorithms.

We then naturally turn to theoretically verify the interesting phenomenon shown by Fig.1. That is, the aim of our study is to answer the following questions:

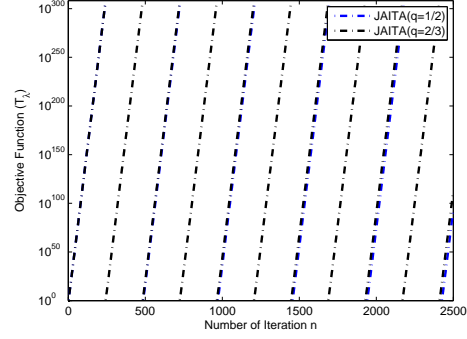
- (Q1) Is the convergence condition of GAITA exactly weaker than that of JAITA?
- (Q2) If the answer of the above question is positive, then what is the applicable range of  $\mu$  for GAITA to guarantee the convergence?

### 1.3. Related Literatures

There are many methods used to solve the  $(l_q\text{LS})$  problem. Some general methods such as those in [1, 2, 7, 9, 10, 14, 16, 19] and references therein and also books [5, 29] do not update



(a) Convergence of GAITA



(b) Divergence of JAITA

Figure 1: An experiment that motivates the use of the Gauss-Seidel scheme. (a) The trends of the objective function sequences, i.e.,  $\{T_\lambda(x^n)\}$  of GAITA for different  $q$ . (b) The trends of the objective function sequences of JAITA for different  $q$ .

the iterations by using the Gauss-Seidel scheme. In [9], the subsequential convergence of the iterative thresholding algorithm for  $(l_q\text{LS})$  with an arbitrary  $q \in (0, 1)$  and further the global convergence for  $(l_q\text{LS})$  with a rational  $q$  have been verified under the condition  $0 < \mu < \|A\|_2^{-2}$ . In [1], the global convergence of the iterative thresholding algorithm for  $(l_q\text{LS})$  with an arbitrary  $q$  has been justified under the same condition. Besides these general methods, there are several specific iterative thresholding algorithms for solving  $(l_q\text{LS})$  with a specific  $q$  such as *hard* for  $l_0$  [8], *soft* for  $l_1$  [17] and *half* for  $l_{1/2}$  [38]. Under the same condition, all these specific iterative thresholding algorithms converge to a stationary point.

Another tightly related class of algorithms is the block coordinate descent (BCD) algorithm. BCD has been numerously used in many applications. Its original form, block coordinate minimization (BCM) can date back to the 1950's [21]. The main idea of BCM is to update a block by minimizing the original objective with respect to that block. Its convergence was extensively studied under many different cases (cf. [20], [31], [34], [37] and the references therein). In [25], the convergence rate of BCM was developed under the strong convexity assumption for the multi-block case, and in [4], its convergence rate was established under the general convexity assumption for the two-block case. Besides BCM, the block coordinate gradient descent (BCGD) method was also largely studied (cf. [35]). Different from BCM, BCGD updates a block via taking a block gradient step, which is equivalent to minimizing a certain prox-linear approximation of the objective. Its global convergence was justified under the assumptions of

the so-called local Lipschitzian error bound and the convexity of the non-differentiable part of the objective. In [28], a randomized block coordinate descent (RBCD) method was proposed. RBCD randomly chooses the block to update with positive probability at each iteration and is not essentially cyclic. The weak convergence was established in [28], [32], while there is no strong convergence result for RBCD.

One important subclass of BCD is the cyclic coordinate descent (CCD) algorithm. The CCD algorithm updates the iterations by the cyclic coordinate updating rule. The work [37] used cyclic updates of a fixed order and supposes block-wise convexity. In [27], a CCD algorithm was proposed for a class of non-convex penalized least squares problems. However, both [27] and [34] did not consider the CCD algorithm for the  $(l_q\text{LS})$  problem. In [18], a CCD algorithm was implemented for solving the  $(l_1\text{LS})$  problem. Its convergence can be shown by referring to [34]. In [33], the  $l_0\text{LS-CD}$  algorithm was proposed for the  $(l_0\text{LS})$  problem, and its convergence to a local minimizer was also shown under certain conditions. Recently, Marjanovic and Solo [26] proposed a cyclic descent algorithm (called  $l_q\text{CD}$ ) for the  $(l_q\text{LS})$  problem with  $0 < q < 1$  and  $A$  being column-normalized, i.e.,  $\|A_i\|_2 = 1$ ,  $i = 1, 2, \dots, N$ , where  $A_i$  is the  $i$ -th column of  $A$ . They proved the subsequential convergence and further the convergence to a local minimizer under the so-called scalable restricted isometry property (SRIP) in [26]. In the perspective of the iterative form,  $l_q\text{CD}$  is a special case of GAITA with  $A$  being column-normalized and  $\mu = 1$ .

#### 1.4. Contributions

The main contribution of this paper is to present the convergence analysis of GAITA for solving the  $(l_q\text{LS})$  problem. The finite step convergence of the support set and sign can be verified under the condition that the step size  $\mu$  is less than  $\frac{1}{\max_i \|A_i\|_2^2}$  (see Theorem 3.7). It means that the support sets and signs of the sequence  $\{x^n\}$  generated by GAITA certainly converge and remain the same within the finite iterations. Such property is very important since it can bring a possible way to construct an auxiliary sequence, which lies in a special subspace and has the same convergence behavior of the original sequence  $\{x^n\}$ . Then with the help of the Kurdyka-Łojasiewicz property (See Appendix G) of  $T_\lambda$ , we can verify the global convergence of GAITA under the same condition, i.e.,  $0 < \mu < \frac{1}{\max_i \|A_i\|_2^2}$  (See Theorem 3.10). It can be noted that this condition is generally weaker than that of JAITA (i.e.,  $0 < \mu < \|A\|_2^{-2}$ ) [1]. This gives positive answers to question (Q1) and (Q2). The improvement on the convergence condition is commonly very important. It may improve not only the rate-of-convergence but

also the applicability of GAITA as compared with JAITA. Furthermore, we can also justify that the proposed algorithm converges to a local minimizer under certain a second-order condition (See Theorem 3.11). More specifically, let  $x^*$  be the limit point and  $I$  be its support set. Then the condition can be described as:  $A_I^T A_I + \lambda q(q-1)\Lambda(x_I^*)$  is positive definite, where  $A_I$  represents the submatrix of  $A$  with column restricted to the index set  $I$ ,  $x_I^*$  is the subvector of  $x^*$  restricted to  $I$ , and  $\Lambda(x_I^*)$  is a diagonal matrix with  $(|x_i^*|^{q-2})_{i \in I}$  as the diagonal vector. Besides this condition, we also give another two sufficient conditions to guarantee that the limit point is a local minimizer. The effectiveness, particularly, the faster convergence and weaker convergence condition of GAITA than JAITA have also been demonstrated by a series of numerical experiments. All these results show that utilizing the Gauss-Seidel iteration in ITA for solving  $(l_q\text{LS})$  is feasible and efficient.

### 1.5. Organization

The remainder of this paper is organized as follows. Some preliminaries are given in section 2. In section 3, we give the convergence analysis of GAITA. In section 4, a series of simulations are implemented to demonstrate the effectiveness of the proposed algorithm. We conclude this paper in section 5, and present all proofs in Appendix.

## 2. Preliminaries

In this section, we present some preliminaries, which serve as the basis of the convergence analysis in the next section.

With the definition of the thresholding function (1.2), we can define a new operator  $G_{\mu, \lambda \| \cdot \|_q^q}(\cdot)$  as

$$G_{\mu, \lambda \| \cdot \|_q^q}(x) = \text{Prox}_{\mu, \lambda \| \cdot \|_q^q}(x - \mu A^T(Ax - y)) \quad (2.1)$$

for any  $x \in \mathbf{R}^N$ . We denote  $\mathcal{F}_q$  as the fixed point set of the operator  $G_{\mu, \lambda \| \cdot \|_q^q}$ , i.e.,

$$\mathcal{F}_q = \{x : x = G_{\mu, \lambda \| \cdot \|_q^q}(x)\}. \quad (2.2)$$

By the definition of  $\text{Prox}_{\mu, \lambda \| \cdot \|_q^q}$ , a type of optimality conditions of the  $(l_q\text{LS})$  problem has been derived in [26].

**Lemma 2.1.** (Theorem 3 in [26]). *Given a point  $x^*$ , define the support set of  $x^*$  as  $\text{Supp}(x^*) = \{i : x_i^* \neq 0\}$ , then  $x^* \in \mathcal{F}_q$  if and only if the following three conditions hold.*

- (a) For  $i \in \text{Supp}(x^*)$ ,  $|x_i^*| \geq \eta_{\mu, q}$ .



- (b) For  $i \in \text{Supp}(x^*)$ ,  $A_i^T(Ax^* - y) + \lambda q \text{sgn}(x_i^*)|x_i^*|^{q-1} = 0$ .  
(c) For  $i \in \text{Supp}(x^*)^c$ ,  $|A_i^T(Ax^* - y)| \leq \tau_{\mu,q}/\mu$ .

We call  $x^*$  a **stationary point** of the  $(l_q\text{LS})$  problem henceforth if it satisfies the optimality conditions in Lemma 2.1. Similarly, according to the definition of the operator  $\text{prox}_{\mu,\lambda|\cdot|^q}(\cdot)$ , (1.6), and the updating rule of GAITA (1.9)-(1.12), we can claim that  $x^{n+1}$  satisfies the following property.

**Property 2.2.** *Given the current iterate  $x^n$  ( $n \in \mathbf{N}$ ), the index set  $i$  is determined via (1.9), then  $x_i^{n+1}$  satisfies either*

- (a)  $x_i^{n+1} = 0$ , or,  
(b)  $|x_i^{n+1}| \geq \eta_{\mu,q}$  and also satisfies the following equation

$$\begin{aligned} & A_i^T(Ax^{n+1} - y) + \lambda q \text{sgn}(x_i^{n+1})|x_i^{n+1}|^{q-1} \\ &= \left(\frac{1}{\mu} - A_i^T A_i\right)(x_i^n - x_i^{n+1}). \end{aligned} \quad (2.3)$$

that is,  $\nabla_i T_\lambda(x^{n+1}) = \left(\frac{1}{\mu} - A_i^T A_i\right)(x_i^n - x_i^{n+1})$ , where  $\nabla_i T_\lambda(x^{n+1})$  represents the gradient of  $T_\lambda$  with respect to the  $i$ -th coordinate at the point  $x^{n+1}$ .

As shown by Property 2.2, the coordinate-wise gradient of  $T_\lambda$  with respect to the  $i$ -th coordinate at  $x^{n+1}$  is not exact zero but with a relative error. This property can be easily derived from the definition of  $\text{prox}_{\mu,\lambda|\cdot|^q}(\cdot)$  and the specific iterative form of GAITA. More specifically, according to (1.6) and (1.11), it holds obviously either  $x_i^{n+1} = 0$  or  $|x_i^{n+1}| \geq \eta_{\mu,q}$ . Moreover, when  $|x_i^{n+1}| \geq \eta_{\mu,q}$ , according to (1.3),  $x_i^{n+1}$  is a minimizer of the optimization problem (1.3) with  $z = z_i^n$ . Therefore,  $x_i^{n+1}$  should satisfy the following optimality condition

$$x_i^{n+1} - z_i^n + \lambda \mu q \text{sgn}(x_i^{n+1})|x_i^{n+1}|^{q-1} = 0. \quad (2.4)$$

Plugging (1.10) into (2.4) gives

$$A_i^T(Ax^{n+1} - y) + \lambda q \text{sgn}(x_i^{n+1})|x_i^{n+1}|^{q-1} = \frac{1}{\mu}(x_i^n - x_i^{n+1}) - A_i^T A(x^n - x^{n+1}). \quad (2.5)$$

Combining (1.12) and (2.5) implies (2.3).

### 3. Convergence Analysis

In this section, we first show the subsequential convergence of GAITA, then prove its global convergence, and further justify that the algorithm can converge to a local minimizer.

### 3.1. Subsequential Convergence

To aid the description, we show that the sequence  $\{T_\lambda(x^n)\}$  satisfies the sufficient decrease property [6] at first.

**Property 3.1.** *Let  $\{x^n\}$  be a sequence generated by GAITA. Assume that  $0 < \mu < L_{\max}^{-1}$ , then*

$$T_\lambda(x^n) - T_\lambda(x^{n+1}) \geq \frac{1}{2} \left( \frac{1}{\mu} - L_{\max} \right) \|x^n - x^{n+1}\|_2^2, \quad \forall n \in \mathbf{N},$$

where  $L_{\max} = \max_i \|A_i\|_2^2$ .

The proof of this property is presented in Appendix 5.1. From this property, we can claim that the objective sequence  $\{T_\lambda(x^n)\}$  converges since it is lower bounded by 0, that is, GAITA is weakly convergent. Furthermore, if the initialization of the sequence is bounded, then based on Property 3.1, it can easily derive the following boundedness and asymptotically regular properties of the sequence.

**Property 3.2.** *Let  $\{x^n\}$  be a sequence generated by GAITA with a bounded initialization. Assume  $0 < \mu < L_{\max}^{-1}$ , then  $\{x^n\}$  is bounded for any  $n \in \mathbf{N}$ , and*

$$\sum_{k=0}^n \|x^{k+1} - x^k\|_2^2 \leq \frac{2\mu}{1 - \mu L_{\max}} T_\lambda(x^0),$$

and also

$$\|x^n - x^{n+1}\|_2 \rightarrow 0, \quad \text{as } n \rightarrow +\infty.$$

The boundedness of  $\{x^n\}$  is mainly due to the sufficient decrease property, the coercivity of  $T_\lambda$  and the boundedness assumption of the initialization. While the asymptotic regular property is mainly due to the sufficient decrease property and the boundedness of the initialization. From Properties 3.1 and 3.2, we can justify the subsequential convergence of GAITA.

**Theorem 3.3.** *Let  $\{x^n\}$  be a sequence generated by GAITA with a bounded initialization. Assume that  $0 < \mu < L_{\max}^{-1}$ , then the sequence  $\{x^n\}$  has a convergent subsequence. Moreover, let  $\mathcal{L}$  be the set of the limit points of  $\{x^n\}$ , then  $\mathcal{L}$  is closed and connected.*

The proof of this theorem is presented in Appendix 5.2. This theorem only shows the subsequential convergence of GAITA. Moreover, we note that  $\mathcal{L}$  might not be a set of isolated points. Due to this, it becomes challenging to justify the global convergence of GAITA [41]. More specifically, there are still two questions on the convergence of the proposed algorithm:

- (a) When does the algorithm converge globally? Under what conditions, GAITA converges strongly in the sense that the whole sequence generated, regardless of the initial point, is convergent.

- (b) Where does the algorithm converge? Does the algorithm converge to a global minimizer or more practically, a local minimizer due to the non-convexity of the optimization problem?

### 3.2. Global Convergence

In this subsection, we will focus on answering the first question proposed in the end of the last subsection. More specifically, we will show that the whole sequence  $\{x^n\}$  generated by GAITA converges as long as the step size  $\mu \in (0, L_{\max}^{-1})$ .

Given the current iteration  $x^n$ , we define the descent function as

$$\Delta(x^n, x^{n+1}) = T_\lambda(x^n) - T_\lambda(x^{n+1}). \quad (3.1)$$

Note that  $x^n$  and  $x^{n+1}$  differ only in their  $i$ -th coefficient which is determined by (1.9). From now on, if not stated, it is assumed that  $x_i^{n+1}$  is given by (1.11) and  $i$  is given by (1.9). The following lemma presents an important property of the descent function.

**Lemma 3.4.** *Let  $\{x^n\}$  be a sequence generated by GAITA. Assume that  $0 < \mu < L_{\max}^{-1}$ , then*

$$\Delta(x^n, x^{n+1}) = 0 \text{ if and only if } x_i^{n+1} = x_i^n.$$

The proof of this lemma is obvious. On one hand, if  $x_i^{n+1} = x_i^n$ , then  $x^{n+1} = x^n$ , and thus  $\Delta(x^n, x^{n+1}) = 0$ . On the other hand, if  $\Delta(x^n, x^{n+1}) = 0$ , then Property 3.1 implies  $x^{n+1} = x^n$  and thus,  $x_i^{n+1} = x_i^n$ .

Moreover, similar to Theorem 10 in [26], we can claim that the mapping  $\mathcal{T}(\cdot, \cdot)$  is a closed mapping, shown as follows.

**Lemma 3.5.**  *$\mathcal{T}(\cdot, \cdot)$  is a closed mapping, i.e., assume*

- (a)  $x_i^n \rightarrow x_i^*$  as  $n \rightarrow \infty$ ;
- (b)  $x_i^{n+1} \rightarrow x_i^{**}$  as  $n \rightarrow \infty$ , where  $x_i^{n+1} = \mathcal{T}(z_i^n, x_i^n)$ .

*Then  $x_i^{**} = \mathcal{T}(z_i^*, x_i^*)$ , where  $z_i^* = x_i^* - \mu A_i^T(Ax^* - y)$ .*

The proof is the essentially the same as that of Theorem 10 in [26]. The only difference is that  $prox_{\mu, \lambda|\cdot|^q}$  is discontinuous at  $\tau_{\mu, q}$  while  $prox_{1, \lambda|\cdot|^q}$  is discontinuous at  $\tau_{1, q}$ . Therefore, the closedness of the operator  $\mathcal{T}(\cdot, \cdot)$  can not be changed after introducing a stepsize  $\mu$ . The following theorem shows that any limit point of the sequence  $\{x^n\}$  is a stationary point of the  $(l_q\text{LS})$  problem.

**Theorem 3.6.** *Let  $\{x^n\}$  be a sequence generated by GAITA with a bounded initialization, and  $\mathcal{L}$  be its limit point set. Suppose that  $0 < \mu < L_{\max}^{-1}$ , then  $\mathcal{L} \subseteq \mathcal{F}_q$ .*

The proof of this theorem is similar to that of Theorem 5 in [26]. For the completion, we provide the proof in Appendix 5.3.

In the following theorem, we justify the finite step convergence of the support sets and signs of the sequence  $\{x^n\}$ , that is, the support sets and signs of  $\{x^n\}$  will converge and remain the same within a finite iterations.

**Theorem 3.7.** *Let  $\{x^n\}$  be a sequence generated by GAITA with a bounded initialization. Assume that  $0 < \mu < L_{\max}^{-1}$  and  $x^*$  is any limit point of  $\{x^n\}$ , then there exists a sufficiently large positive integer  $n^* > N$  such that when  $n > n^*$ , it holds*

- (a) either  $x_j^n = 0$  or  $|x_j^n| \geq \eta_{\mu,q}$  for  $j = 1, 2, \dots, N$ ;
- (b)  $I^n = I$ ;
- (c)  $\text{sgn}(x^n) = \text{sgn}(x^*)$ ,

where  $I^n = \text{Supp}(x^n) = \{i : |x_i^n| \neq 0, i = 1, 2, \dots, N\}$  and  $I = \text{Supp}(x^*)$ .

The proof of this theorem is shown in Appendix 5.4. From this theorem, it can be observed that when  $n$  is sufficiently large, the generated sequence  $\{x^n\}$  as well as its limit points will lie in the same subspace  $S \subset \mathbf{R}^N$ , which has some special structure. Due to this, it brings a possible way to construct an auxiliary sequence that has the same convergence behavior of the original sequence  $\{x^n\}$ . Thus, we only need to verify the convergence of the constructed auxiliary sequence instead of  $\{x^n\}$ . The construction of the auxiliary sequence is a bit standard and is motivated by [41]. To be detailed, the sequence can be constructed according to the following procedure.

- (a) Let  $n_0 = j_0 N > n^*$  for some positive integer  $j_0$ . Then we can define a new sequence  $\{\hat{x}^n\}$  with  $\hat{x}^n = x^{n_0+n}$  for  $n \in \mathbf{N}$ . It is obvious that  $\{\hat{x}^n\}$  has the same convergence behavior with  $\{x^n\}$ . Moreover, it can be noted from Theorem 3.7 that all the support sets and signs of  $\{\hat{x}^n\}$  are the same.
- (b) Denote  $I$  as the convergent support set of the sequence  $\{x^n\}$ . Let  $K$  be the number of elements of  $I$ . Without loss of generality, we assume

$$1 \leq I(1) < I(2) < \dots < I(K) \leq N.$$

According to the updating rule (1.9)-(1.12) of GAITA, we can observe that many successive iterations of  $\{\hat{x}^n\}$  are the same. Thus, we can merge these successive iterations into a single iteration. Moreover, the updating rule of the index is cyclic and thus periodic. As

a consequence, the merging procedure can be repeated periodically. Formally, we consider such a periodic subsequence with  $N$ -length of  $\{\hat{x}^n\}$ , i.e.,

$$\{\hat{x}^{jN+I(1)}, \hat{x}^{jN+I(1)+1}, \dots, \hat{x}^{jN+I(1)+N-1}\}$$

for  $j \in \mathbf{N}$ . Then for any  $j \in \mathbf{N}$ , we emerge the  $N$ -length sequence  $\{\hat{x}^{jN+I(1)}, \dots, \hat{x}^{jN+I(1)+N-1}\}$  into a new  $K$ -length sequence  $\{\bar{x}^{jK+1}, \bar{x}^{jK+2}, \dots, \bar{x}^{jK+K}\}$  with the rule

$$\{\hat{x}^{jN+I(i)}, \dots, \hat{x}^{jN+I(i+1)-1}\} \mapsto \bar{x}^{jK+i},$$

with  $\bar{x}^{jK+i} = \hat{x}^{jN+I(i)}$  for  $i = 1, 2, \dots, K$ , since  $\hat{x}^{jN+I(i)+k} = \hat{x}^{jN+I(i)}$  for  $k = 1, \dots, I(i+1) - I(i) - 1$ . Moreover, we emerge the first  $I(1)$  iterations of  $\{\hat{x}^n\}$  into  $\bar{x}^0$ , i.e.,

$$\{\hat{x}^0, \dots, \hat{x}^{I(1)-1}\} \mapsto \bar{x}^0,$$

with  $\bar{x}^0 = \hat{x}^0$ , since these iterations keep invariant and are equal to  $\hat{x}^0$ . After this procedure, we obtain a new sequence  $\{\bar{x}^n\}$  with  $n = jK + i$ ,  $i = 0, \dots, K - 1$  and  $j \in \mathbf{N}$ . It can be observed that such an emerging procedure keeps the convergence behavior of  $\{\bar{x}^n\}$  the same as that of  $\{\hat{x}^n\}$  and  $\{x^n\}$ .

(c) Furthermore, for the index set  $I$ , we define a projection  $P_I$  as

$$P_I : \mathbf{R}^N \rightarrow \mathbf{R}^K, P_I x = x_I, \forall x \in \mathbf{R}^N,$$

where  $x_I$  represents the subvector of  $x$  restricted to the index set  $I$ . With this projection, a new sequence  $\{u^n\}$  is constructed such that

$$u^n = P_I \bar{x}^n,$$

for  $n \in \mathbf{N}$ . As we can observe that  $u^n$  keeps all the non-zero elements of  $\bar{x}^n$  while gets rid of its zero elements. Moreover, this operation can not change the convergence behavior of  $\{\bar{x}^n\}$  and  $\{u^n\}$ . Therefore, the convergence behavior of  $\{u^n\}$  is the same as  $\{x^n\}$ .

After the construction procedure (a)-(c), we get a new sequence  $\{u^n\}$ . In the following, we will prove the convergence of  $\{x^n\}$  via justifying the convergence of  $\{u^n\}$ . Let

$$\mathcal{U} = \{u^* : u^* = P_I x^*, \forall x^* \in \mathcal{L}\}.$$

Then  $\mathcal{U}$  is the corresponding limit point set of  $\{u^n\}$ . Furthermore, we define a new function  $T$  as follows:

$$T : \mathbf{R}^K \rightarrow \mathbf{R}, T(u) = T_\lambda(P_I^T u), \forall u \in \mathbf{R}^K, \quad (3.2)$$

where  $P_I^T$  denotes the transpose of the projection  $P_I$ , and is defined as

$$P_I^T : \mathbf{R}^K \rightarrow \mathbf{R}^N, (P_I^T u)_I = u, (P_I^T u)_{I^c} = 0, \forall u \in \mathbf{R}^K.$$

Here  $I^c$  represents the complementary set of  $I$ , i.e.,  $I^c = \{1, 2, \dots, N\} \setminus I$ ,  $(P_I^T u)_I$  and  $(P_I^T u)_{I^c}$  represent the subvectors of  $P_I^T u$  restricted to  $I$  and  $I^c$ , respectively. Let  $B = A_I$ , where  $A_I$  denotes the submatrix of  $A$  restricted to the index set  $I$ . Thus,

$$T(u) = \frac{1}{2} \|Bu - y\|_2^2 + \lambda \|u\|_q^q.$$

After the construction procedure (a)-(c), we can observe that the following properties still hold for  $\{u^n\}$ .

**Lemma 3.8.** *The sequence  $\{u^n\}$  possesses the following properties:*

- (a)  $\{u^n\}$  is updated via the following cyclic rule. Given the current iteration  $u^n$ , only the  $i$ -th coordinate will be updated while the other coordinate coefficients will be fixed at the next iteration, i.e.,

$$u_i^{n+1} = \mathcal{T}(v_i^n, u_i^n), \quad (3.3)$$

and

$$u_j^{n+1} = u_j^n, \text{ for } j \neq i, \quad (3.4)$$

where  $i$  is determined by

$$i = \begin{cases} K & \text{if } 0 \equiv (n+1) \bmod K \\ (n+1) \bmod K, & \text{otherwise} \end{cases}, \quad (3.5)$$

and

$$v_i^n = u_i^n - \mu B_i^T (Bu^n - y), \quad (3.6)$$

- (b) According to the updating rules (3.3)-(3.6), for  $n \geq K$ , there exist two positive integers  $1 \leq i_0 \leq K$  and  $j_0 \geq 1$  such that  $n = j_0 K + i_0$  and

$$u_j^n = \begin{cases} u_j^{n-(i_0-j)}, & \text{if } 1 \leq j \leq i_0 \\ u_j^{n-K-(i_0-j)}, & \text{if } i_0 + 1 \leq j \leq K \end{cases}. \quad (3.7)$$

- (c) For any  $n \in \mathbf{N}$ ,

$$u^n \in \mathbf{R}_{\eta_{\mu,q}^c}^K,$$

where  $\mathbf{R}_{\eta_{\mu,q}^c}$  represents a one-dimensional real subspace, which is defined as

$$\mathbf{R}_{\eta_{\mu,q}^c} = \mathbf{R} \setminus (-\eta_{\mu,q}, \eta_{\mu,q}).$$

- (d) Given  $u^n$ , if  $i$  is determined by (3.5), then  $u_i^{n+1}$  satisfies the following equation

$$\begin{aligned} & B_i^T (Bu^{n+1} - y) + \lambda q \operatorname{sgn}(u_i^{n+1}) |u_i^{n+1}|^{q-1} \\ &= \left( \frac{1}{\mu} - B_i^T B_i \right) (u_i^n - u_i^{n+1}). \end{aligned} \quad (3.8)$$

That is,

$$\nabla_i T(u^{n+1}) = \left(\frac{1}{\mu} - B_i^T B_i\right)(u_i^n - u_i^{n+1}),$$

where  $\nabla_i T(u^{n+1})$  represents the gradient of  $T(\cdot)$  with respect to the  $i$ -th coordinate at the point  $u^{n+1}$ .

(e)  $\{u^n\}$  satisfies the following sufficient decrease condition:

$$T(u^n) - T(u^{n+1}) \geq a \|u^n - u^{n+1}\|_2^2,$$

for  $n \in \mathbf{N}$ , where  $a = \frac{1}{2}(\frac{1}{\mu} - L_{\max})$ .

(f)  $\|u^{n+1} - u^n\|_2 \rightarrow 0$ , as  $n \rightarrow \infty$ .

It can be observed that the properties of  $\{u^n\}$  listed in Lemma 3.8 are some direct extensions of those of  $\{x^n\}$ . More specifically, Lemma 3.8(a) can be derived by updating rules (1.9)-(1.12) and the construction procedure. Lemma 3.8(b) is obtained directly by the cyclic updating rule. Lemma 3.8(c) and (d) can be derived by Property 2.2(b) and the updating rules (3.3)-(3.6). Lemma 3.8(e) can be obtained by Property 3.1 and the definition of  $T$  (3.2). Lemma 3.8(f) can be directly derived by Property 3.2. Besides Lemma 3.8, the following lemma shows that the gradient sequence  $\{\nabla T(u^n)\}$  satisfies the so-called relative error condition [1], which is critical to the justification of the convergence of  $\{u^k\}$ .

**Lemma 3.9.** *When  $n \geq K - 1$ ,  $\nabla T(u^{n+1})$  satisfies*

$$\|\nabla T(u^{n+1})\|_2 \leq b \|u^{n+1} - u^n\|_2,$$

where  $b = (\frac{1}{\mu} + K\delta)\sqrt{K}$ , with

$$\delta = \max_{i,j=1,2,\dots,K} |B_i^T B_j|.$$

The proof of this lemma is given in Appendix 5.5. From Lemma 3.8 (e), the sequence  $\{u^n\}$  satisfies the sufficient decrease condition with respect to  $T$ , and by Lemma 3.9,  $\{u^n\}$  satisfies the relative error condition, and also by the continuity of  $T$ ,  $\{u^n\}$  satisfies the so-called continuity condition. Furthermore, according to [1] (p. 122), we know that the function

$$T(u) = \frac{1}{2} \|Bu - y\|_2^2 + \lambda \|u\|_q^q$$

is a **Kurdyka-Łojasiewicz (KL)** function (see Appendix 5.7). Thus, according to Theorem 2.9 in [1],  $\{u^n\}$  is convergent. As a consequence, we can claim the convergence of  $\{x^n\}$  as shown in the following theorem.

**Theorem 3.10.** *Let  $\{x^n\}$  be a sequence generated by GAITA with a bounded initialization. Assume that  $0 < \mu < L_{\max}^{-1}$ , then  $\{x^n\}$  converges to a stationary point.*

According to [1], the convergence condition of JAITA when applied to the  $(l_q\text{LS})$  problem is  $0 < \mu < \|A\|_2^{-2}$ . It can be noted that  $\max_i \|A_i\|_2^2 \leq \|A\|_2^2$ , and hence the condition in Theorem 3.10 is generally weaker than that of JAITA. Moreover, as shown by Fig. 1, such improvement on the convergence condition is solid and essential in the sense that there exists a step size  $\mu \in (\|A\|_2^{-2}, L_{\max}^{-1})$  such that JAITA certainly diverges while GAITA definitely converges with this given step size.

Suppose that  $A$  is column-normalized, i.e.,  $\|A_i\|_2 = 1$  for any  $i$ , then  $L_{\max} = 1$ , and thus the condition of GAITA becomes  $0 < \mu < 1$ . In this setting, if further  $\mu = 1$ , then GAITA reduces to the  $l_q\text{CD}$  algorithm [26] in the perspective of the iterative form. However, only the subsequential convergence of the  $l_q\text{CD}$  algorithm can be claimed in [26] if there is no additional requirement of  $A$ . Compared with the  $l_q\text{CD}$  algorithm, there are mainly two significant improvements. The first one is that we extend the column-normalized  $A$  to a general  $A$ . Such extension on the model can improve the flexibility and applicability of GAITA. The second one, and also the more important one is that the global convergence of GAITA can be established. It gives a solidly theoretical guarantee to the use of GAITA.

### 3.3. Convergence to A Local Minimizer

In this subsection, we mainly answer the second open question proposed in the end of the subsection 3.1. More specifically, we will justify that GAITA converges to a local minimizer of the  $(l_q\text{LS})$  problem under certain conditions.

**Theorem 3.11.** *Let  $\{x^n\}$  be a sequence generated by GAITA with a bounded initialization. Assume that  $0 < \mu < L_{\max}^{-1}$ , and  $x^*$  is the convergent point of  $\{x^n\}$ . Let  $I = \text{Supp}(x^*)$ , and  $K = \|x^*\|_0$ . Then  $x^*$  is a (strictly) local minimizer of  $T_\lambda$  if the following condition holds:*

$$A_I^T A_I + \lambda q(q-1)\Lambda(x_I^*) \succ 0, \quad (3.9)$$

where  $A_I$  represents the submatrix of  $A$  with column restricted to  $I$ ,  $x_I^*$  is the subvector of  $x$  restricted to  $I$ ,  $\Lambda(x_I^*) \in \mathbf{R}^{K \times K}$  is a diagonal matrix with  $(|x_i^*|^{q-2})_{i \in I}$  as the diagonal vector, and  $M \succ 0$  represents that  $M$  is positive definite for any matrix  $M$ .

The proof of this theorem is given in Appendix 5.6. Intuitively, under the condition of Theorem 3.11, it follows that the principle submatrix of the Hessian matrix of  $T_\lambda$  at  $x^*$  restricted to the index set  $I$  is positive definite. Moreover, by Lemma 2.1 (b), the following first-order optimality condition holds

$$A_I^T (Ax^* - y) + \lambda \phi_1(x_I^*) = 0,$$



where  $\phi_1(x_I^*) = (q \operatorname{sgn}(x_{i_1}^*)|x_{i_1}^*|^{q-1}, \dots, q \operatorname{sgn}(x_{i_K}^*)|x_{i_K}^*|^{q-1})^T$ , and  $i_j \in I, j = 1, \dots, K$ . These two conditions imply that the second-order optimality conditions hold at  $x^* = (x_I^*, 0)$ . For any sufficiently small  $h$ , let  $x^h = x^* + h$ , then

$$\begin{aligned} T_\lambda(x^h) &= \frac{1}{2} \|A_I x_I^h - y + A_{I^c} h_{I^c}\|_2^2 + \lambda \sum_{i \in I} |x_i^h|^q + \lambda \sum_{i \in I^c} |h_i|^q \\ &= \frac{1}{2} \|A_I x_I^h - y\|_2^2 + \lambda \sum_{i \in I} |x_i^h|^q \\ &\quad + \frac{1}{2} \|A_{I^c} h_{I^c}\|_2^2 + \langle h_{I^c}, A_{I^c}^T (A_I x_I^h - y) \rangle + \lambda \sum_{i \in I^c} |h_i|^q. \end{aligned}$$

Denote  $T_{I^c} = \frac{1}{2} \|A_{I^c} h_{I^c}\|_2^2 + \langle h_{I^c}, A_{I^c}^T (A_I x_I^h - y) \rangle + \lambda \sum_{i \in I^c} |h_i|^q$ . Then

$$\begin{aligned} T_\lambda(x^h) &\geq T_\lambda(x^*) + T_{I^c} \\ &\geq T_\lambda(x^*) + \frac{1}{2} \|A_{I^c} h_{I^c}\|_2^2 + \sum_{i \in I^c} (\lambda |h_i|^q - \|A_{I^c}^T (A_I x_I^h - y)\|_\infty |h_i|), \end{aligned}$$

where the first inequality holds for the optimality at  $x^* = (x_I^*, 0)$  and thus,  $\frac{1}{2} \|A_I x_I^h - y\|_2^2 + \lambda \sum_{i \in I} |x_i^h|^q \geq T_\lambda(x^*)$ . It can be observed that if  $h_{I^c}$  is sufficiently small, then the last part of the above inequality should be nonnegative. Therefore,  $x^*$  should be a local minimizer.

Furthermore, we can drive another two sufficient conditions via taking advantage of the specific form of the threshold value (1.8). Let  $e = \min_{i \in I} |x_i^*|$ . Note that

$$\lambda_{\min}(A_I^T A_I + \lambda q(q-1) \Lambda(x_I^*)) \geq \lambda_{\min}(A_I^T A_I) + \lambda q(q-1) e^{q-2},$$

where  $\lambda_{\min}(M)$  represents the minimal eigenvalue of a given matrix  $M$ . Thus, if

$$\lambda_{\min}(A_I^T A_I) > 0 \text{ and } 0 < \lambda < \frac{\lambda_{\min}(A_I^T A_I) e^{2-q}}{q(1-q)}, \quad (3.10)$$

then the condition of Theorem 3.11 holds naturally.

Moreover, by (1.8), it holds

$$e \geq \eta_{\mu, q} = (2\lambda\mu(1-q))^{\frac{1}{2-q}}. \quad (3.11)$$

Hence, if  $\frac{\lambda_{\min}(A_I^T A_I)}{\max_i \|A_i\|_2^2} > \frac{q}{2}$  and

$$\frac{q}{2\lambda_{\min}(A_I^T A_I)} < \mu < \frac{1}{\max_i \|A_i\|_2^2}, \quad (3.12)$$

then the condition (3.10) holds and thus (3.9) also holds. According to the above analysis, we can easily obtain the following theorem.

**Theorem 3.12.** *Let  $\{x^n\}$  be a sequence generated by GAITA with a bounded initialization. Assume that  $0 < \mu < L_{\max}^{-1}$ , and  $x^*$  is the convergent point of  $\{x^n\}$ . Let  $I = \text{Supp}(x^*)$ ,  $K = \|x^*\|_0$ , and  $e = \min_{i \in I} |x_i^*|$ . Then  $x^*$  is a (strictly) local minimizer of  $T_\lambda$  if either of the two following conditions satisfies:*

- (a)  $\lambda_{\min}(A_I^T A_I) > 0, 0 < \lambda < \frac{\lambda_{\min}(A_I^T A_I)e^{2-q}}{q(1-q)}$ ;
- (b)  $\frac{\lambda_{\min}(A_I^T A_I)}{\max_i \|A_i\|_2^2} > \frac{q}{2}, \frac{q}{2\lambda_{\min}(A_I^T A_I)} < \mu < \frac{1}{\max_i \|A_i\|_2^2}$ .

Intuitively, the condition (a) in Theorem 3.12 means that if the smooth part of the  $(l_q\text{LS})$  problem is strictly convex and the regularization parameter is sufficiently small, then the convexity of  $T_\lambda$  at  $x^*$  can be guaranteed by the convexity of the smooth part. Suppose that  $A$  is column-normalized, i.e.,  $\|A_i\|_2 = 1$  for any  $i$ , then the condition (b) in Theorem 3.12 intuitively implies that if the smooth part of the  $(l_q\text{LS})$  problem is strongly convex, then the local convexity of  $T_\lambda$  at  $x^*$  can be guaranteed as long as the step size  $\mu$  is chosen appropriately. Similar conditions are also derived for the iterative *half* thresholding algorithm for solution to the  $(l_q\text{LS})$  problem with  $q = 1/2$  (See Theorems 1 and 2 in [40]). However, the conditions in this theorem are a little weaker than those in [40].

In [26], the convergence of the  $l_q\text{CD}$  algorithm to a local minimizer is justified under a certain scalable restricted isometry property (SRIP). SRIP is defined as follows.

**Definition 3.13. (SRIP [3]).** *We say  $A$  has the SRIP( $p, \phi, \alpha$ ) if there exist  $\nu_\phi, \gamma_\phi > 0$  satisfying  $\gamma_\phi/\nu_\phi < \alpha$  such that*

$$\nu_\phi \|x\|_2 \leq \|Ax\|_2 \leq \gamma_\phi \|x\|_2$$

*holds for every  $x \in B_p(\phi) := \{x : \|x\|_p^p \leq \phi\}$ , and  $\|\cdot\|_p^p := \|\cdot\|_0$  for  $p = 0$ .*

Roughly speaking,  $\nu_\phi$  and  $\gamma_\phi$  can be viewed as some type of the minimal and maximal singular values of  $A$ , respectively. Thus, SRIP essentially indicates that  $A$  possesses a good condition number. With the definition of SRIP, [26] demonstrates that if  $A$  has the SRIP( $p, \phi, \alpha$ ) with some  $p \geq 0$ , then for any  $0 < q < q^*$  (where  $q^* := \min\{1, 2/\alpha^2\}$ ), the  $l_q\text{CD}$  algorithm converges to a local minimizer. Particularly, when  $\alpha = \sqrt{2}$ , that is,  $\gamma_\phi/\nu_\phi < \sqrt{2}$ , then the  $l_q\text{CD}$  algorithm converges to a local minimizer for any  $0 < q < 1$ . In other words, if

$$0 < q < \min\left\{1, \frac{2\nu_\phi^2}{\gamma_\phi^2}\right\}, \quad (3.13)$$

then the  $l_q\text{CD}$  algorithm can converge to a local minimizer. It can be seen from Theorem 3.12 that the condition (b) is equivalent to

$$0 < q < \min\left\{1, \frac{2\lambda_{\min}(A_I^T A_I)}{\max_i \|A_i\|_2^2}\right\}. \quad (3.14)$$

It is generally hard to compare the conditions (3.13) and (3.14) directly. However, if  $p = 0$ , then SRIP may reduce to the standard restricted isometry property (RIP), and in this case, if further  $\phi = K$  (where  $K$  is the cardinality of the support set of  $x^*$ ), then

$$\lambda_{\min}(A_I^T A_I) \geq \nu_K^2 \text{ and } \max_i \|A_i\|_2^2 \leq \gamma_K^2.$$

Therefore,

$$\frac{\lambda_{\min}(A_I^T A_I)}{\max_i \|A_i\|_2^2} \geq \frac{\nu_K^2}{\gamma_K^2},$$

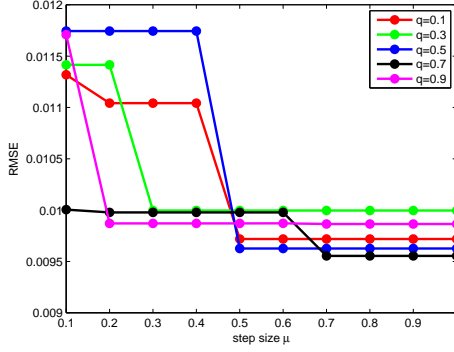
which implies that our conditions for convergence to a local minimizer are generally weaker than that of the  $l_q$ CD algorithm in terms of the SRIP.

## 4. Numerical Experiments

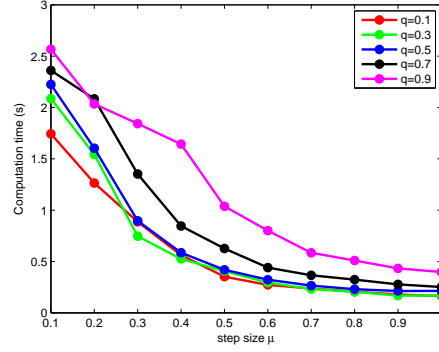
In this section, we demonstrate the effects of the algorithmic parameters on the performance of GAITA. Particularly, we will mainly focus on the effect of the step size parameter, while the effects of the regularization parameter  $\lambda$  and  $q$  can be referred to [26]. Moreover, a series of experiments are conducted to show the faster convergence as well as the weaker convergence condition of GAITA as compared with JAITA.

### 4.1. On effect of $\mu$

For this purpose, we considered the performance of GAITA for the sparse signal recovery problem, i.e.,  $y = Ax + \epsilon$ , where  $x \in \mathbf{R}^N$  was an unknown sparse signal,  $A \in \mathbf{R}^{m \times N}$  was the measurement matrix,  $y \in \mathbf{R}^m$  was the corresponding measurement vector,  $\epsilon$  was the noise and generally  $m < N$ . The aim of this problem was to recover the sparse signal  $x$  from  $y$ . In these experiments, we set  $m = 250$ ,  $N = 500$  and  $k^* = 15$ , where  $k^*$  was the sparsity level of the original sparse signal. The original sparse signal  $x^*$  was generated randomly according to the standard Gaussian distribution.  $A$  was of dimension  $m \times N = 250 \times 500$  with Gaussian  $\mathcal{N}(0, 1/250)$  i.i.d. entries and was preprocessed via column-normalization, i.e.,  $\|A_i\|_2 = 1$  for any  $i$ . The observation  $y$  was added with 30 dB noise. With these settings, the convergence condition of GAITA becomes  $0 < \mu < 1$ . To justify the effect of the step size, we varied  $\mu$  from 0 to 1, and considered different  $q$ , that is,  $q = 0.1, 0.3, 0.5, 0.7, 0.9$ . The terminal rule of GAITA was set as the recovery mean square error (RMSE)  $\frac{\|x^n - x^*\|_2}{\|x^*\|_2}$  less than a given precision  $tol$  (in this case,  $tol = 10^{-2}$ ). The regularization parameter  $\lambda$  was set as 0.009 and fixed for all experiments. The experiment results are shown in Fig. 2.



(a) Recovery Error



(b) Computational Time

Figure 2: Experiment for the justification of the effect of the step size parameter  $\mu$  on the performance of GAITA with different  $q$ . (a) The trends of recovery error of GAITA with different  $q$ . (b) The trends of the computational time of GAITA with different  $q$ .

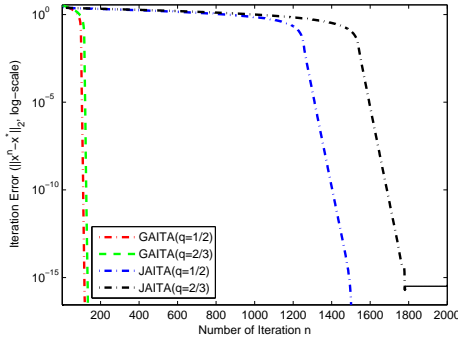
From Fig. 2, we can observe that the step size parameter  $\mu$  has almost no influence on the recovery quality of the proposed algorithm (as shown in Fig. 2(a)) while it significantly affects the time efficiency of the proposed algorithm (as shown in Fig. 2(b)). Basically, we can claim that the larger step size implies the faster convergence. This coincides with the common consensus. Therefore, in practice, we suggest a larger step size like  $0.95/L_{\max}$  for GAITA.

## 4.2. Comparison with JAITA

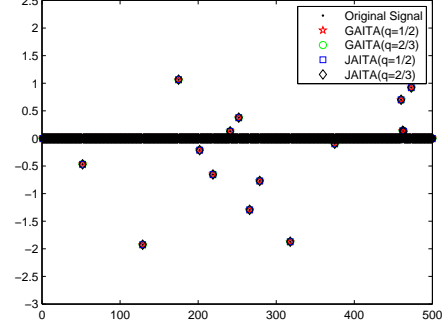
### 4.2.1. Faster Convergence

We conducted an experiment to demonstrate the faster convergence of GAITA as compared with JAITA [38], [41]. For this purpose, given a sparse signal  $x$  with dimension  $N = 500$  and sparsity  $k^* = 15$ , shown as in Fig. 3(b), we considered the signal recovery problem through the observation  $y = Ax$ , where the measurement matrix  $A$  and the original sparse signal  $x$  were generated according to the same way in section 4.1. We then applied GAITA and JAITA to the  $(l_q\text{LS})$  problem with two different  $q$ , that is,  $q = 1/2$  and  $2/3$ , respectively. In both cases, we took  $\lambda = 0.001$ ,  $\mu = \frac{0.95}{\max_i \|A_i\|_2^2} (= 0.95)$  for GAITA and  $\mu = 0.99\|A\|_2^{-2} (= 0.1676)$  for JAITA. Moreover, the initial guess was 0. For better comparison, we took every  $N$  inner iterations of GAITA as one iteration since in this case, all coordinates were updated only once. The experiment results are reported in Fig. 3.

It can be seen from Fig. 3(a) how the iteration error ( $\|x^n - x^*\|_2$ ) varies. It can be observed that GAITA converges much more rapidly than JAITA in both cases. As shown in Fig. 3(a),



(a) Iteration error



(b) Recovery signal

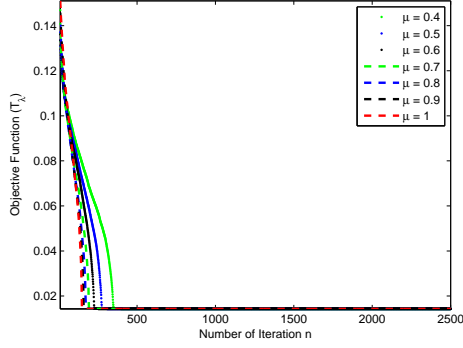
Figure 3: Experiment for convergence rate. (a) The trend of iteration error, i.e.,  $\|x^n - x^*\|_2$ . (b) Recovery signal. The Recovery MSEs of the four cases, that is, GAITA ( $q = 1/2$ ), GAITA ( $q = 2/3$ ), JAITA ( $q = 1/2$ ) and JAITA ( $q = 2/3$ ) are  $2.06 \times 10^{-8}$ ,  $5.14 \times 10^{-9}$ ,  $2.12 \times 10^{-8}$  and  $5.28 \times 10^{-9}$ , respectively.

the numbers of iterations needed for GAITA are about 150 in both cases, while much more iterations are required for JAITA (say, about 1500 and 1700 iterations for  $q = 1/2$  and  $2/3$ , respectively). As justified in [40], [41], JAITA possesses the eventually linear convergence rate, that is, JAITA will converge linearly after certain iterations. From Fig. 3(a), the similar eventually linear convergence rate of GAITA can be observed. Also, compared with JAITA, much fewer iterations are required to start such a linear decay. Moreover, Fig. 3(b) shows that the original sparse signal is recovered by both GAITA and JAITA with very high accuracies. This experiment clearly shows the faster convergence as well as eventually linear convergence rate properties of GAITA.

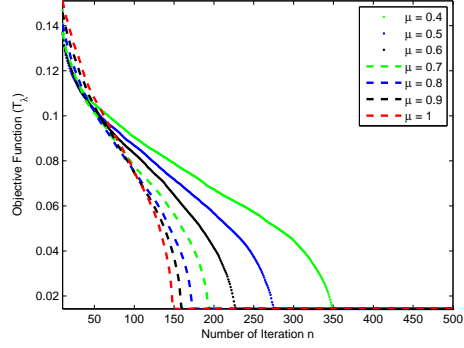
#### 4.2.2. Weaker Condition

We conducted a set of experiments to demonstrate the convergence condition of GAITA is weaker than that of JAITA. The experiment setting was the same as the above subsection. We then applied GAITA and JAITA to the ( $l_q$ LS) problem with  $q = 1/2$ . In this setting, the theoretical condition for convergence of JAITA is  $\mu \in (0, 0.1759)$  while the associated condition of GAITA is  $\mu \in (0, 1)$ . We used different  $\mu$  (i.e.,  $\mu = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1$ ) for both GAITA and JAITA. The figures of the objective function sequences are shown in Fig. 4.

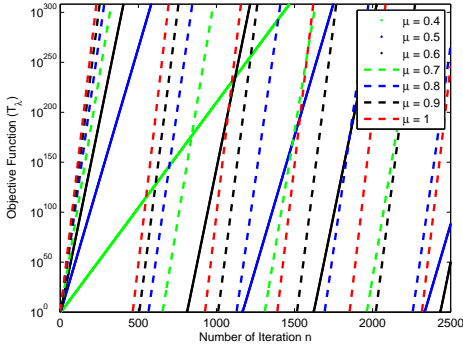
From Fig. 4, the objective sequences of JAITA diverge for all  $\mu$ , while those of GAITA are certainly convergent. These can be observed detailedly from Fig. 4(b) and (d), respectively. By



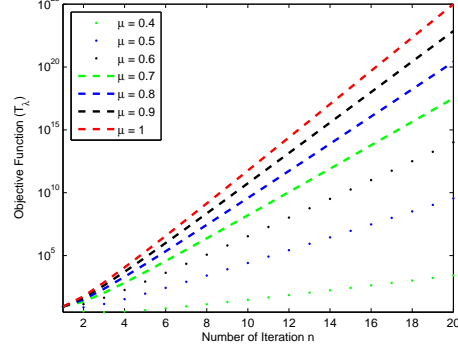
(a) Convergence of GAITA



(b) Detail of GAITA



(c) Divergence of JAITA



(d) Detail of JAITA

Figure 4: An experiment that verifies the weaker convergence condition of GAITA as compared with JAITA. (a) The trends of the objective function sequences, i.e.,  $\{T_\lambda(x^n)\}$  of GAITA with different  $\mu$ . (b) The detail trends of the objective function sequences of GAITA with different  $\mu$ . (c) The trends of the objective function sequences of JAITA with different  $\mu$ . (d) The detail trends of the objective function sequences of JAITA with different  $\mu$ . The regularization parameter  $\lambda$  was taken as 0.001 in all cases.

Fig. 4(a), the objective sequences of GAITA can converge fast within about 400 iterations in all cases, while those sequences of JAITA diverge rapidly as shown by Fig. 4(d). When  $\mu = 1$  and  $A$  is column-normalized, GAITA is reduced to the  $l_q$ CD method. Fig. 4(a) and (b) show the objective sequence of the  $l_q$ CD method is convergent, which can be actually guaranteed by Property 3.1. It implies that the  $l_q$ CD method is weakly convergent as justified in [26]. However, different from GAITA, the global convergence of the  $l_q$ CD method has not been justified if there is no additional condition.

## 5. Conclusion

In this paper, we focused on utilizing the Gauss-Seidel iteration rule to the iterative thresholding algorithm for the non-convex  $l_q$  regularized least squares regression problem and developed a new algorithm called GAITA. The main contributions of this paper are the establishment of the convergence of the proposed algorithm. In summary, we have verified that

- (i) GAITA has the finite step convergence of the support set and sign as long as the step size  $0 < \mu < 1/L_{\max}$ . It means that the support sets and signs of the sequence generated by GAITA can converge and remain the same within finite iterations.
- (ii) Under the same condition, the global convergence of GAITA can be justified. Compared with JAITA like *half* algorithm for  $l_{1/2}$  regularization, the convergence condition of GAITA is weaker than that of JAITA (i.e.,  $0 < \mu < \|A\|_2^{-2}$ ).
- (iii) If certain a second-order condition is satisfied at the limit point, then the limit point can indeed be a local minimizer. Thus, under these conditions, the proposed algorithm converges to a local minimizer.
- (iv) Several numerical experiments are implemented to demonstrate the effectiveness of GAITA, particularly, the expected faster convergence and desired weaker convergence condition than JAITA. Also, the similar eventually linear convergence rate of GAITA can be observed. However, such rate of convergence property of GAITA has not been justified in the current paper, and we will study this in the future work.

When it comes to parallel implementation, however, GAITA could have certain disadvantages because variables that depend on each other can only be updated sequentially.

## Appendix

Most of proofs and the description of Kurdyka-Łojasiewicz inequality are presented in Appendix.

### 5.1. Proof of Property 3.1

**Proof.** Given the current iteration  $x^n$ , let the coefficient index  $i$  be determined according to (1.9). According to (1.3) and (1.11),

$$x_i^{n+1} \in \arg \min_{v \in \mathbf{R}} \left\{ \frac{|z_i^n - v|^2}{2} + \lambda \mu |v|^q \right\},$$

where  $z_i^n = x_i^n - \mu A_i^T(Ax^n - y)$ . Then it implies

$$\frac{1}{2}|\mu A_i^T(Ax^n - y)|^2 + \lambda\mu|x_i^n|^q \geq \frac{1}{2}|(x_i^{n+1} - x_i^n) + \mu A_i^T(Ax^n - y)|^2 + \lambda\mu|x_i^{n+1}|^q.$$

Some simplifications give

$$\lambda|x_i^n|^q - \lambda|x_i^{n+1}|^q \geq \frac{|x_i^{n+1} - x_i^n|^2}{2\mu} + A_i^T(Ax^n - y)(x_i^{n+1} - x_i^n). \quad (5.1)$$

Moreover, since  $x_j^{n+1} = x_j^n$  for any  $j \neq i$ , (5.1) becomes

$$\lambda\|x^n\|_q^q - \lambda\|x^{n+1}\|_q^q \geq \frac{\|x^{n+1} - x^n\|^2}{2\mu} + \langle Ax^n - y, A(x^{n+1} - x^n) \rangle. \quad (5.2)$$

Adding  $\frac{1}{2}\|Ax^n - y\|_2^2 - \frac{1}{2}\|Ax^{n+1} - y\|_2^2$  to both sides of (5.2) gives

$$\begin{aligned} & T_\lambda(x^n) - T_\lambda(x^{n+1}) \\ & \geq \frac{\|x^{n+1} - x^n\|^2}{2\mu} - \frac{1}{2}\|A(x^n - x^{n+1})\|_2^2 \\ & = \frac{\|x^{n+1} - x^n\|^2}{2\mu} - \frac{1}{2}(A_i^T A_i)\|x^n - x^{n+1}\|_2^2 \\ & \geq \frac{1}{2}\left(\frac{1}{\mu} - L_{\max}\right)\|x^n - x^{n+1}\|_2^2, \end{aligned} \quad (5.3)$$

where the first equality holds for

$$\|A(x^n - x^{n+1})\|_2^2 = (A_i^T A_i)|x_i^n - x_i^{n+1}|^2 = (A_i^T A_i)\|x^n - x^{n+1}\|_2^2,$$

and the second inequality holds for  $A_i^T A_i \leq L_{\max}$ . ■

### 5.2. Proof of Theorem 3.3

**Proof.** By Property 3.1, we know that  $\{T_\lambda(x^n)\}$  is a decreasing and lower-bounded sequence, thus,  $\{T_\lambda(x^n)\}$  is convergent. Denote the convergent value of  $\{T_\lambda(x^n)\}$  as  $T^*$ . Moreover, by Property 3.2,  $\{x^n\}$  is bounded, and also by the continuity of  $T_\lambda(\cdot)$ , there exists a subsequence of  $\{x^n\}$ ,  $\{x^{n_j}\}$  converging to some point  $x^*$ , which satisfies  $T_\lambda(x^*) = T^*$ .

Furthermore, by Property 3.2 and Ostrowski's result (Theorem 26.1, p. 173) [30], the limit point set  $\mathcal{L}$  of the sequence  $\{x^n\}$  is closed and connected. ■

### 5.3. Proof of Theorem 3.6

**Proof.** Since the sequence  $\{x^n\}$  is bounded, then it has limit points. Let  $x^* \in \mathcal{L}$ . We now focus on the  $i$ -th coefficient of the sequence with  $n = n(i) = jN + i - 1$ , where  $i = 1, 2, \dots, N$



and  $j = 0, 1, \dots$ . However, here, we simply use  $n$  by which we mean  $n(i)$ . Now there exists a subsequence  $\{x^{n_1}, x^{n_2}, \dots\}$  such that

$$\{x^{n_1}, x^{n_2}, \dots\} \rightarrow x^* \text{ and } \{x_i^{n_1}, x_i^{n_2}, \dots\} \rightarrow x_i^*. \quad (5.4)$$

Moreover, since the sequence  $\{x^{n_1+1}, x^{n_2+1}, \dots\}$  is also bounded, thus, it also has limit points. Denoting one of these by  $x^{**}$ , then there exists a subsequence  $\{x^{l_1+1}, x^{l_2+1}, \dots\}$  such that

$$\{x^{l_1+1}, x^{l_2+1}, \dots\} \rightarrow x^{**} \text{ and } \{x_i^{l_1+1}, x_i^{l_2+1}, \dots\} \rightarrow x_i^{**}, \quad (5.5)$$

where  $\{l_1, l_2, \dots\} \subset \{n_1, n_2, \dots\}$ . In this case, it holds

$$\{x^{l_1}, x^{l_2}, \dots\} \rightarrow x^* \text{ and } \{x_i^{l_1}, x_i^{l_2}, \dots\} \rightarrow x_i^*, \quad (5.6)$$

since it is a subsequence of (5.4). From (1.10) and (5.6), we have

$$z_i^{l_j} \rightarrow z_i^* \text{ as } j \rightarrow \infty.$$

Thus, by Lemma 3.5, it holds

$$x_i^{**} = \mathcal{T}(z_i^*, x_i^*). \quad (5.7)$$

Moreover, by (5.5), (5.6) and (1.12), it holds

$$x_j^* = x_j^{**} \text{ for } j \neq i. \quad (5.8)$$

In the following, by the continuity of  $T_\lambda(\cdot)$  and thus the continuity of  $\Delta(\cdot, \cdot)$  with respect to its arguments, it holds

$$\Delta(x^{l_j}, x^{l_j+1}) \rightarrow \Delta(x^*, x^{**}).$$

Moreover, since the sequence  $\{T_\lambda(x^n)\}$  is convergent, then

$$\Delta(x^{l_j}, x^{l_j+1}) = T_\lambda(x^{l_j}) - T_\lambda(x^{l_j+1}) \rightarrow 0 \text{ as } j \rightarrow \infty,$$

which implies

$$\Delta(x^*, x^{**}) = 0.$$

Furthermore, by Lemma 3.4, and (5.7)-(5.8), it holds

$$x_i^{**} = x_i^*. \quad (5.9)$$

Combining (5.7) and (5.9), we have

$$x_i^* = \mathcal{T}(z_i^*, x_i^*). \quad (5.10)$$

Since  $i$  is arbitrary, we have that (5.10) holds for all  $i \in \{1, \dots, N\}$ . It implies that  $x^*$  is a fixed point of  $G_{\mu, \lambda, \|\cdot\|_q^q}(\cdot)$ , that is,  $x^* \in \mathcal{F}_q$ . Similarly, since  $x^* \in \mathcal{L}$  is also arbitrary, therefore,  $\mathcal{L} \subset \mathcal{F}_q$ . Consequently, we complete the proof of this theorem. ■

#### 5.4. Proof of Theorem 3.7

**Proof.** We can note that all the coefficient indices will be updated at least one time when  $n > N$ . By Property 2.2, once the index  $i$  is updated at the  $n$ -th iteration, then the coefficient  $x_i^n$  satisfies:

$$\text{either } x_i^n = 0 \text{ or } |x_i^n| \geq \eta_{\mu, q}.$$

Thus, Theorem 3.7(a) holds.

In the following, we prove Theorem 3.7(b) and (c). By the assumption of Theorem 3.7, there exists a subsequence  $\{x^{n_j}\}$  converges to  $x^*$ , i.e.,

$$x^{n_j} \rightarrow x^* \text{ as } j \rightarrow \infty. \quad (5.11)$$

Thus, there exists a sufficiently large positive integer  $j_0$  such that  $\|x^{n_j} - x^*\|_2 < \eta_{\mu, q}$  when  $j \geq j_0$ . Moreover, by Property 3.2, there also exists a sufficiently large positive integer  $n^* > N$  such that  $\|x^n - x^{n+1}\|_2 < \eta_{\mu, q}$  when  $n > n^*$ . Without loss of generality, we let  $n^* = n_{j_0}$ . In the following, we first prove that  $I^n = I$  and  $\text{sgn}(x^n) = \text{sgn}(x^*)$  whenever  $n > n^*$ .

In order to prove  $I^n = I$ , we first show that  $I^{n_j} = I$  when  $j \geq j_0$  and then verify that  $I^{n+1} = I^n$  when  $n > n^*$ . We now prove by contradiction that  $I^{n_j} = I$  whenever  $j \geq j_0$ . Assume this is not the case, namely, that  $I^{n_j} \neq I$ . Then we easily derive a contradiction through distinguishing the following two possible cases:

*Case 1:*  $I^{n_j} \neq I$  and  $I^{n_j} \cap I \subset I^{n_j}$ . In this case, then there exists an  $i_{n_j}$  such that  $i_{n_j} \in I^{n_j} \setminus I$ . By Theorem 3.7(a), it then implies

$$\|x^{n_j} - x^*\|_2 \geq |x_{i_{n_j}}^{n_j}| \geq \min_{i \in I^{n_j}} |x_i^{n_j}| \geq \eta_{\mu, q},$$

which contradicts to  $\|x^{n_j} - x^*\|_2 < \eta_{\mu, q}$ .

*Case 2:*  $I^{n_j} \neq I$  and  $I^{n_j} \cap I = I^{n_j}$ . In this case, it is obvious that  $I^{n_j} \subset I$ . Thus, there exists an  $i^*$  such that  $i^* \in I \setminus I^{n_j}$ . By Lemma 2.1(a), we still have

$$\|x^{n_j} - x^*\|_2 \geq |x_{i^*}^{n_j}| \geq \min_{i \in I} |x_i^*| \geq \eta_{\mu, q},$$

and it contradicts to  $\|x^{n_j} - x^*\|_2 < \eta_{\mu, q}$ .

Thus we have justified that  $I^{n_j} = I$  when  $j \geq j_0$ . Similarly, it can be also claimed that  $I^{n+1} = I^n$  whenever  $n > n^*$ . Therefore, whenever  $n > n^*$ , it holds  $I^n = I$ .

As  $I^n = I$  when  $n > n^*$ , it suffices to test that  $\text{sgn}(x_i^{(n)}) = \text{sgn}(x_i^*)$  for any  $i \in I$ . Similar to the first part of proof, we will first check that  $\text{sgn}(x_i^{n_j}) = \text{sgn}(x_i^*)$ , and then  $\text{sgn}(x_i^{n+1}) = \text{sgn}(x_i^n)$  for any  $i \in I$  by contradiction. We now prove  $\text{sgn}(x_i^{n_j}) = \text{sgn}(x_i^*)$  for any  $i \in I$ . Assume this is not the case. Then there exists an  $i^* \in I$  such that  $\text{sgn}(x_{i^*}^{n_j}) \neq \text{sgn}(x_{i^*}^*)$ , and hence,

$$\text{sgn}(x_{i^*}^{n_j})\text{sgn}(x_{i^*}^*) = -1.$$

From Lemma 2.1(a) and Theorem 3.7(a), it then implies

$$\begin{aligned} \|x^{n_j} - x^*\|_2 &\geq |x_{i^*}^{n_j} - x_{i^*}^*| = |x_{i^*}^{n_j}| + |x_{i^*}^*| \\ &\geq \min_{i \in I} \{|x_i^{n_j}| + |x_i^*|\} \geq 2\eta_{\mu,q}, \end{aligned}$$

contradicting again to  $\|x^{n_j} - x^*\|_2 < \eta_{\mu,q}$ . This contradiction shows  $\text{sgn}(x^{n_j}) = \text{sgn}(x^*)$ . Similarly, we can also show that  $\text{sgn}(x^{n+1}) = \text{sgn}(x^n)$  whenever  $n > n^*$ . Therefore,  $\text{sgn}(x^n) = \text{sgn}(x^*)$  when  $n > n^*$ .

With this, the proof of Theorem 3.7 is completed. ■

### 5.5. Proof of Lemma 3.9

**Proof.** We assume that  $n+1 = j^*K + i^*$  for some positive integers  $j^* \geq 1$  and  $1 \leq i^* \leq K$ . For simplicity, let

$$i^* = K. \tag{5.12}$$

If not, we can renumber the indices of the coordinates such that (5.12) holds while the iterative sequence  $\{u^n\}$  keeps invariant, since the updating rule (3.5) is cyclic and thus periodic. Such an operation can be described as follows: for each  $n \geq K$ , by Lemma 3.8(b), we know that the coefficients of  $u^n$  are only related to the previous  $K-1$  iterates. Thus, we consider the following a period of the original updating order, i.e.,

$$\{i^* + 1, \dots, K, 1, \dots, i^*\},$$

then we can renumber the above coordinate updating order as

$$\{1', \dots, (K - i^*)', (K - i^* + 1)', \dots, K'\},$$

with

$$j' = \begin{cases} i^* + j, & \text{if } 1 \leq j \leq K - i^* \\ j - (K - i^*), & \text{if } K - i^* < j \leq K \end{cases}.$$

In the following, we will calculate  $\nabla_i T(u^{n+1})$  by a recursive way for  $i = K, K-1, \dots, 1$ . Specifically,

(a) For  $i = K$ , by Lemma 3.8(d), it holds

$$\nabla_K T(u^{n+1}) = \left(\frac{1}{\mu} - B_K^T B_K\right)(u_K^n - u_K^{n+1}). \quad (5.13)$$

For any  $i = K-1, K-2, \dots, 1$ ,

$$\nabla_i T(u^{n+1}) = B_i^T (Bu^{n+1} - y) + \lambda q \operatorname{sgn}(u_i^{n+1}) |u_i^{n+1}|^{q-1},$$

and  $u_i^{n+1} = u_i^n$ . Therefore, for  $i = K-1, K-2, \dots, 1$ ,

$$\nabla_i T(u^{n+1}) = \nabla_i T(u^n) + B_i^T B_K (u_K^{n+1} - u_K^n). \quad (5.14)$$

(b) For  $i = K-1$ , since  $n = j^* K + (K-1)$ , then by Lemma 3.8(d) again, it holds

$$\nabla_{K-1} T(u^n) = \left(\frac{1}{\mu} - B_{K-1}^T B_{K-1}\right)(u_{K-1}^{n-1} - u_{K-1}^n). \quad (5.15)$$

By Lemma 3.8(b), it implies

$$u_{K-1}^{n-1} = u_{K-1}^{n+1}.$$

Thus,

$$\nabla_{K-1} T(u^n) = \left(\frac{1}{\mu} - B_{K-1}^T B_{K-1}\right)(u_{K-1}^{n+1} - u_{K-1}^n). \quad (5.16)$$

Combing (5.14) with (5.16),

$$\nabla_{K-1} T(u^{n+1}) = \left(\frac{1}{\mu} - B_{K-1}^T B_{K-1}\right)(u_{K-1}^{n+1} - u_{K-1}^n) + B_{K-1}^T B_K (u_K^{n+1} - u_K^n). \quad (5.17)$$

Similarly to (5.14), for  $i = K-2, K-3, \dots, 1$ , we have

$$\nabla_i T(u^n) = \nabla_i T(u^{n-1}) + B_i^T B_{K-2} (u_{K-2}^n - u_{K-2}^{n-1}). \quad (5.18)$$

(c) For any  $i = K - j$  with  $0 \leq j \leq K - 1$ , by a recursive way, we have

$$\begin{aligned}
& \nabla_{K-j} T(u^{n+1}) \\
&= \nabla_{K-j} T(u^n) + B_{K-j}^T B_K (u_K^{n+1} - u_K^n) \\
&= \nabla_{K-j} T(u^{n-1}) + B_{K-j}^T \sum_{k=0}^1 B_{K-k} (u_{K-k}^{n+1-k} - u_{K-k}^{n-k}) \\
&= \dots \\
&= \nabla_{K-j} T(u^{n-j+1}) + B_{K-j}^T \sum_{k=0}^{j-1} B_{K-k} (u_{K-k}^{n+1-k} - u_{K-k}^{n-k}). \tag{5.19}
\end{aligned}$$

Moreover, Lemma 3.8(d) gives

$$\nabla_{K-j} T(u^{n-j+1}) = \left( \frac{1}{\mu} - B_{K-j}^T B_{K-j} \right) (u_{K-j}^{n-j} - u_{K-j}^{n-j+1}). \tag{5.20}$$

Plugging (5.20) into (5.19), it holds

$$\nabla_{K-j} T(u^{n+1}) = \frac{1}{\mu} (u_{K-j}^{n-j} - u_{K-j}^{n-j+1}) + \sum_{k=0}^j B_{K-j}^T B_{K-k} (u_{K-k}^{n+1-k} - u_{K-k}^{n-k}), \tag{5.21}$$

for  $j = 0, 1, \dots, K - 1$ . Furthermore, by Lemma 3.8(b), it implies

$$u_{K-k}^{n+1-k} = u_{K-k}^{n+1}$$

and

$$u_{K-k}^{n-k} = u_{K-k}^n$$

for  $0 \leq k \leq K - 1$ . Therefore, (5.21) becomes

$$\nabla_{K-j} T(u^{n+1}) = \frac{1}{\mu} (u_{K-j}^n - u_{K-j}^{n+1}) + \sum_{k=0}^j B_{K-j}^T B_{K-k} (u_{K-k}^{n+1} - u_{K-k}^n), \tag{5.22}$$

for  $j = 0, 1, \dots, K - 1$ .

Furthermore, by (5.22), it implies

$$\begin{aligned}
|\nabla_{K-j} T(u^{n+1})| &\leq \frac{1}{\mu} |u_{K-j}^n - u_{K-j}^{n+1}| + \sum_{k=0}^j (|B_{K-j}^T B_{K-k}| \cdot |u_{K-k}^{n+1} - u_{K-k}^n|) \\
&\leq \frac{1}{\mu} |u_{K-j}^n - u_{K-j}^{n+1}| + \delta \|u^{n+1} - u^n\|_1, \tag{5.23}
\end{aligned}$$

for  $j = 0, 1, \dots, K - 1$ , where the second inequality holds for

$$\delta = \max_{i,j=1,\dots,K} |B_i^T B_j|$$

and

$$\sum_{k=0}^j |u_{K-k}^{n+1} - u_{K-k}^n| \leq \|u^{n+1} - u^n\|_1.$$

Summing  $|\nabla_{K-j} T(u^{n+1})|$  with respect to  $j$  gives

$$\begin{aligned} \|\nabla T(u^{n+1})\|_1 &\leq \frac{1}{\mu} \|u^{n+1} - u^n\|_1 + K\delta \|u^{n+1} - u^n\|_1 \\ &\leq \left(\frac{1}{\mu} + K\delta\right) \sqrt{K} \|u^{n+1} - u^n\|_2, \end{aligned} \quad (5.24)$$

where the second inequality holds for the norm inequality between 1-norm and 2-norm, that is,

$$\|u\|_2 \leq \|u\|_1 \leq \sqrt{K} \|u\|_2, \quad (5.25)$$

for any  $u \in \mathbf{R}^K$ . Also, combining (5.25) and (5.24) implies

$$\|\nabla T(u^{n+1})\|_2 \leq \left(\frac{1}{\mu} + K\delta\right) \sqrt{K} \|u^{n+1} - u^n\|_2.$$

■

### 5.6. Proof of Theorem 3.11

**Proof.** Let  $F(x) = \frac{1}{2} \|Ax - y\|_2^2$  and

$$\phi_1(x_I^*) = (q \operatorname{sgn}(x_{i_1}^*) |x_{i_1}^*|^{q-1}, \dots, q \operatorname{sgn}(x_{i_K}^*) |x_{i_K}^*|^{q-1})^T,$$

where  $i_j \in I, j = 1, \dots, K$ . By Lemma 2.1(b) we have

$$A_I^T (Ax^* - y) + \lambda \phi_1(x_I^*) = 0. \quad (5.26)$$

This together with the condition of the theorem

$$A_I^T A_I + \lambda q(q-1) \Lambda(x_I^*) \succ 0$$

imply that the second-order optimality conditions hold at  $x^* = (x_I^*, 0)$ . For sufficiently small vector  $h$ , we denote  $x_h^* = (x_I^* + h_I, 0)$ . It then follows

$$F(x_h^*) + \lambda \sum_{i \in I} |x_i^* + h_i|^q \geq F(x^*) + \lambda \sum_{i \in I} |x_i^*|^q. \quad (5.27)$$

Furthermore, for any  $q \in (0, 1)$ , it obviously holds that

$$t^q > (\|\nabla_{I^c} F(x^*)\|_\infty + 2)t/\lambda,$$

for sufficiently small  $t > 0$ . By this fact and the differentiability of  $F$ , one can observe that for sufficiently small  $h$ , there hold

$$\begin{aligned} F(x^* + h) - F(x_h^*) + \lambda \sum_{i \in I^c} |h_i|^q &= \nabla_{I^c}^T F(x^*) h_{I^c} + \lambda \sum_{i \in I^c} |h_i|^q + o(h_{I^c}) \\ &\geq \sum_{i \in I^c} (\|\nabla_{I^c} F(x^*)\|_\infty - [\nabla_{I^c} F(x^*)]_i + 1) |h_i| \geq 0. \end{aligned} \quad (5.28)$$

Summing up the above two equalities (5.27)-(5.28), one has that for all sufficiently small  $h$ ,

$$T_\lambda(x^* + h) - T_\lambda(x^*) \geq 0, \quad (5.29)$$

and hence  $x^*$  is a local minimizer.

Actually, we can observe that when  $h \neq 0$ , then at least one of the two inequalities (5.27) and (5.28) will hold strictly, which implies that  $x^*$  is a strictly local minimizer. ■

### 5.7. Kurdyka-Lojasiewicz Inequality

- (a) The function  $f : \mathbf{R} \rightarrow \mathbf{R} \cup \{+\infty\}$  is said to have the Kurdyka-Lojasiewicz property at  $x^* \in \text{dom } \partial f$  if there exist  $\eta \in (0, +\infty]$ , a neighborhood  $U$  of  $x^*$  and a continuous concave function  $\varphi : [0, \eta) \rightarrow \mathbf{R}_+$  such that:

- (i)  $\varphi(0) = 0$ ;
- (ii)  $\varphi$  is  $\mathcal{C}^1$  on  $(0, \eta)$ ;
- (iii) for all  $s \in (0, \eta)$ ,  $\varphi'(s) > 0$ ;
- (iv) for all  $x$  in  $U \cap \{x : f(x^*) < f(x) < f(x^*) + \eta\}$ , the Kurdyka-Lojasiewicz inequality holds

$$\varphi'(f(x) - f(x^*)) \text{dist}(0, \partial f(x)) \geq 1. \quad (5.30)$$

- (b) Proper lower semi-continuous functions which satisfy the Kurdyka-Lojasiewicz inequality at each point of  $\text{dom } \partial f$  are called KL functions.

Functions satisfying the KL inequality include real analytic functions, semialgebraic functions and locally strongly convex functions.

## References

- [1] H. Attouch, J. Bolte and B. F. Svaiter, Convergence of descent methods for semi-algebraic and tame problems: proximal algorithms, forward-backward splitting, and regularized Gauss-Seidel methods, *Mathematical Programming, Ser. A*, 137:91-129, 2013.

- [2] A. M. Bagirov, L. Jin, N. Karmita, A. Al Nuaimat, and N. Sultanova, Subgradient method for nonconvex nonsmooth optimization, *Journal of Optimization Theory and applications*, 157: 416-435, 2013.
- [3] A. Beck and M. Teboulle, A linearly convergent algorithm for solving a class of nonconvex/affine feasibility problems, in *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, ser. Springer Optimization and Its Applications. New York, NY, USA: Springer, 2011, pp. 33-48.
- [4] A. Beck and L. Tetruashvili, On the convergence of block coordinate descent type methods, *SIAM Journal on Optimization*, 23: 2037-2060, 2013.
- [5] D. P. Bertsekas, *Nonlinear Programming*, Athena Scientific, September 1999.
- [6] J. V. Burke, Descent methods for composite nondifferentiable optimization problems, *Math. Program.*, 33: 260-279, 1985.
- [7] J. V. Burke, A. S. Lewis, and M. L. Overton, A robust gradient sampling algorithm for nonsmooth, nonconvex optimization, *SIAM Journal on Optimization*, 15:751-779, 2005.
- [8] T. Blumensath and M. E. Davies, Iterative thresholding for sparse approximation, *Journal of Fourier Analysis and Application*, 14 (5): 629-654, 2008.
- [9] K. Bredies, D. A. Lorenz, and S. Reiterer, Minimization of non-smooth, non-convex functionals by iterative thresholding, *Journal of Optimization Theory and Applications*, 165: 78-122, 2015.
- [10] E. J. Candès, M. B. Wakin, and S. P. Boyd, Enhancing sparsity by reweighted  $l_1$  minimization, *Journal of Fourier Analysis and Applications*, 14 (5): 877-905, 2008.
- [11] W. F. Cao, J. Sun, and Z. B. Xu, Fast image deconvolution using closed-form thresholding formulas of  $L_q$  ( $q = 1/2, 2/3$ ) regularization, *Journal of Visual Communication and Image Representation*, 24 (1): 1529-1542, 2013.
- [12] R. Chartrand, Exact reconstruction of sparse signals via nonconvex minimization, *IEEE Signal Processing Letters*, 14 (10): 707-710, 2007.
- [13] R. Chartrand and V. Staneva, Restricted isometry properties and nonconvex compressive sensing, *Inverse Problems*, 24: 1-14, 2008.



- [14] X. Chen, Smoothing methods for nonsmooth, nonconvex minimization, *Mathematical programming*, 134:71-99, 2012.
- [15] P. Combettes and V. Wajs, Signal recovery by proximal forward-backward splitting. *Multiscale Model. Simul.*, 4: 1168-1200, 2005.
- [16] I. Daubechies, R. DeVore, M. Fornasier, and C. S. Gunturk, Iteratively reweighted least squares minimization for sparse recovery, *Communications on Pure and Applied Mathematics*, 63: 1-38, 2010.
- [17] I. Duabechies, M. Defrise, C. De Mol, An iterative thresholding algorithm for linear inverse problems with a sparse constraint, *Communications on Pure and Applied Mathematics*, 57: 1413-1457, 2004.
- [18] J.H. Friedman, T. Hastie, H. Hofling, and R. Tibshirani, Pathwise coordinate optimization, *Ann. Appl. Stat.*, 1 (2): 302-332, 2007.
- [19] A. Fuduli, M. Gaudioso, and G. Giallombardo, Minimizing nonconvex nonsmooth functions via cutting planes and proximity control, *SIAM Journal on Optimization*, 14: 743-756, 2004.
- [20] L. Grippo and M. Sciandrone, Globally convergent block-coordinate techniques for unconstrained optimization, *Optim. Methods Softw.*, 10: 587-637, 1999.
- [21] C. Hildreth, A quadratic programming procedure, *Naval Research Logistics Quarterly*, 4: 79-85, 1957.
- [22] J. Kivinen, Exponentiated gradient versus gradient descent for linear predictors, *Inf. Comput.*, 132 (1): 1-63, 1997.
- [23] D. Krishnan and R. Fergus, Fast image deconvolution using hyperLaplacian priors, *Proc. Adv. Neural Inf. Process. Syst. (NIPS)*, 2009.
- [24] S.B. Lin, J.S. Zeng, J. Fang, and Z.B. Xu, Learning rates of  $l_q$  coefficient regularization learning with Gaussian kernel, *Neural Computation*, 26 (10): 2350-2378 , 2014.
- [25] Z. Q. Luo and P. Tseng, On the convergence of the coordinate descent method for convex differentiable minimization, *J. Optim. Theory Appl.*, 72: 7-35, 1992.

- [26] G. Marjanovic, and V. Solo,  $l_q$  sparsity penalized linear regression with cyclic descent, IEEE Transactions on Signal Processing, 62 (6): 1464-1475, 2014.
- [27] R. Mazumder, J. H. Friedman, and T. Hastie, Sparsenet: Coordinate descent with nonconvex penalties, J. Amer. Statist. Assoc., 106: 1125-1138, 2007.
- [28] Y. Nesterov, Efficiency of coordinate descent methods on huge-scale optimization problems, SIAM Journal on Optimization, 22: 341-362, 2012.
- [29] J. Nocedal and S. J. Wright, Numerical Optimization, Springer Series in Operations Research and Financial Engineering, Springer, New York, second ed., 2006.
- [30] A.M. Ostrowski, Solutions of equations in Euclidean and Banach spaces, New York, NY, USA: Academic, 1973.
- [31] M. Razaviyayn, M. Hong, and Z.Q. Luo, A unified convergence analysis of block successive minimization methods for nonsmooth optimization, SIAM Journal on Optimization, 23: 1126-1153, 2013.
- [32] P. Richtárik and M. Takáč, Iteration complexity of randomized block-coordinate descent methods for minimizing a composite function, Mathematical Programming, 144:1-38, 2014.
- [33] A. J. Seneviratne and V. Solo, On exact denoising, School Elect. Eng. Telecommun., Univ. New South Wales, New South Wales, Australia, Tech. Rep., 2013.
- [34] P. Tseng, Convergence of a block coordinate descent method for nondifferentiable minimization, Journal of Optimization Theory and Applications, 109: 475-494, 2001.
- [35] P. Tseng and S. Yun, A coordinate gradient descent method for nonsmooth separable minimization, Math. Program., 117: 387-423, 2009.
- [36] J. N. Tsitsiklis, A comparison of Jacobi and Gauss-Seidel parallel iterations, Applied Mathematics Letters, 2(2): 167-170, 1989.
- [37] Y. Xu and W. Yin, A block coordinate descent method for regularized multiconvex optimization with applications to nonnegative tensor factorization and completion, SIAM Journal on Imaging Sciences, 6: 1758-1789, 2013.

- [38] Z. B. Xu, X. Y. Chang, F. M. Xu and H. Zhang,  $L_{1/2}$  regularization: a thresholding representation theory and a fast solver, IEEE Transactions on Neural Networks and Learning Systems, 23: 1013-1027, 2012.
- [39] J. S. Zeng, J. Fang and Z. B. Xu, Sparse SAR imaging based on  $L_{1/2}$  regularization, Science China Series F-Information Science, 55: 1755-1775, 2012.
- [40] J. S. Zeng, S. B. Lin, Y. Wang and Z. B. Xu,  $L_{1/2}$  Regularization: convergence of iterative half thresholding algorithm, IEEE Transactions on Signal Processing, 62 (9): 2317-2329, 2014.
- [41] J. S. Zeng, S. B. Lin and Z. B. Xu, Sparse Regularization: Convergence of Iterative Jumping Thresholding Algorithm, arXiv preprint arXiv:1402.5744, 2014.

